Good afternoon. You hear me all right, right? In the previous lecture, we introduced the medical imaging general ideas and the content, and also TA explained to you some basic knowledge of MATLAB programming. So this third lecture, really the first lecture starting about what I call the foundation part. So this is a foundation needed to understand each of the five major medical imaging modalities. X-ray CT, nuclear imaging, MRI, magnetic resonant imaging, ultrasound imaging, and optical imaging. These are five most important imaging modalities widely used in hospitals and clinics. And to understand these imaging modalities, you do need this foundational knowledge. So I try to improve teaching quality by drafting a chapter, several chapters, five chapters, and that's the first part of my intended new textbook on medical imaging. So the new textbook is one of my dreams. I hope to give a consistent right to point presentation as a BME version of the very classic engineering mathematics, Fourier analysis, linear system, signal processing, and so on. You don't want too much to be an electrical engineering major. You don't want too less. What do I mean by too less? And today's lecture is a good example. And do you know the concept of a system? You quickly know. Since you put it together, you call it a system. And what's a linear system? Linear system has two properties. So additivity and homogeneity. And you will show definition. That's done for this lecture. But why we still need two hours explaining things? So that example, appendix listing formulas for Fourier transform, signal processing, matrix sampling are not enough for deeper understanding. Some of you asked me about the syllabus, and strangely, last semester, several students said they couldn't see the syllabus. I already uploaded it to digital measure, and the student cannot see. And I will contact the IT support this afternoon. But I wanted to make sure. And last year's syllabus already posted in LMS. So you show it, and not so much difference, and we have the lecture pretty much the same. I changed it a little bit. So I basically deleted one optical imaging lecture, replaced with what I call machine learning. Nowadays, artificial intelligence, machine learning, deep learning, so hard you've got to know a little bit. And last semester, in graduate level course, I comprised so much classic stuff. And I gave four lectures on medical imaging. Somehow, my evaluation score, I mentioned, reduced from a perfect 5.0, got a full point or something. So I think maybe too much, so I try to be gentle here. Only one lecture give you kind of popular knowledge. What do I mean by machine learning? And also, some examples in this class, and later on, I will show you a little bit machine learning. This is so hard to see. And I personally like it very much. The key ideas behind the syllabus, what do you really care about, about grading policies? You see, we do not have a final. Instead, we have examinations one, two, three. And how we calculate your final grade, this is your concern most. So this is the roadmap, or just the lookup table. And we always, always request class participation. I think for your best learning results, you download the first part, you download my draft book. It's not perfect. And I really enjoy interacting with students. Some of you download the textbook, part one, and you feel some part wrong, or some places you would word it another way, you do it in tracking mode. You can discuss with me in my office hour, and I myself will keep refining. Maybe by the end of this year, the book will be in good shape, but not now. It's still better than nothing. So you have the textbook. So you see complete, unified workflow, how I think about Fourier analysis and so on. You can really derive, understand the key ideas. So that's so important to do, I call it preview. Then classroom teaching is important. You just like to face-to-face how I present the stuff, so you understand the key point. If you skip the class, you read it yourself, the chance is that you can have the same understanding. But I really think that's not the most effective way. We ask you to participate, but I really do not have a good way to keep track. In the future, I can have a camera, so every student just automatically recognizes using machine learning. So that will be a good way. But now I put this 10%, but I assume all of you just attended the class, unless I have solid evidence that some of you missed the class. I may just deduct 10% of all of them, but normally you just have that. On top of the 10% credit, you need to have three examination scores and homework. TA will take care of that. And the lighter score distribution, I can tell you, always consistent. Like last semester, I remember I had about 20 students, four of them got A, four of them got C, and two of them got D, and the rest are in B range. I use plus and minus occasionally. But this kind of distribution I would expect to maintain for this class. So I do not intensely fail you unless you really surprise me. But to get A is not an easy thing. I do not promise to participate in this class. Most of you get A. That's not the case. But I will do tests and greetings in collaboration with TA very fairly. So just learn the material. And after all, medical imaging is important for your family life and for your RPI education. So today's lecture is simple. I talk about the system. And in the book draft, you see the table of contents. OK, so this is the first chapter you can read. If you read about 10 pages, you pretty much understand. And you listen to me carefully, you pretty much understand. And the chapter goes in a very balanced way. And each chapter, I would have three major aspects organized as section one, two, three. And the section four remarks and some discussions, interesting point. I put some thought. I think quite relevant. And today's lecture is our system. So we talk about general system, just too complicated. But we then focus on linear system. Linear system is linear, just like linear function. It's straightforward. To be a linear system, we ask two properties. You will see additivity and homogeneity. And you can combine them together as the so-called superposition principle. So that's the one line, very clear. But I will explain a little bit in deeper depth. And the two properties, additivity, homogeneity, why you need two properties. And the water condition, from one requirement, you can derive the other. And the water condition, you cannot do so. In other words, are these two requirements independent or not? So I gave some explanation, just inspiring you to think deeply. And the third section, talking about a nonlinear system. Any system that is not linear is nonlinear. You have a whole lot of nonlinear systems. So it's just out of line. So you've got some rough idea. Then pay attention to the rest of the content. So system and function are not so much different. So we have been learning mathematics. So function is a very familiar concept. What is function? Google. In Google, you have multiple definitions about function. But basically, function is about a mapping. So a mathematical relation such that each element in a given side, what you call the domain of function, is associated with an element in another domain called a range. So this is just a mapping relationship. And the system is somehow also do the mapping. So you have input and you have output. And the input into the system, the system is a collection of multiple elements. They are interconnected. They interact together. And it could be social system, a physiological system, or some mechanical system. Could be anything. So function and system are rather general concepts. So we have this concept. And we want to narrow it down to define. So we can derive a quantitative relationship. That can be very important for engineering practice. So this is an illustration about the concept of a function. So you have domain, so anything, so a number. And you pick up in the domain. Then through this functional relationship or map, and you know this particular element corresponds to another element in the range domain. So this is straightforward. But this concept is very important. So I put a red diamond here. So whenever you see a red diamond, that is a key point for you to remember. On the other hand, you see the green button. That's some nice things to know. And not necessarily required. And this is undergraduate level. Interdisciplinary Department, BME. So we are not pure mathematics. But still, we need to know some deeper stuff. Whenever you see green things, you feel relaxed. So just curious to know what's beyond the engineering requirement. But this red diamond really shows something you have to have a very clear idea. So functional relationship is important. And then we have multiple examples, like a sinusoidal function, like linear function in one dimensional, in two dimensional. And then you can use recursive relationship to define so-called fractal measure. Those of you not so much familiar with fractal geometry, and you could check Google. The fractal thing is a very, very fancy idea. But anyway, at least you know regular function. And then we have so-called Taylor expansion. So you can just express an arbitrary continuous well-behaved function as a summation of many, many terms. Looking at this expression, the first term is constant. This is the functional value at a point x0. So if you just ignore other terms, you pretty much just represent a continuous function as a constant. That's a very rough first zero-order approximation. If you want to do better, you'll find the first derivative at this point. Then you just assume there is a line going through this point with the same slope as you computed here. So you use a right line to represent this arbitrary function. And if you still feel the accuracy is not sufficient, you could just say you do a quadratic curve through this point. That quadratic curve happens to have the same functional value, same first-order derivative, and the same second-order derivative. So if you just take these three terms, you do second-order derivative. You will find this quadratic expression happens to have the same second-order derivative at this point. So you can keep adding these high-order terms. You'll get a better and a better approximation under certain conditions. You need to make sure these series, infinitely many terms added together got converted to the real functional value. So this is not an arbitrary thing. Under moderate conditions, you can do so. And for mathematical analysis and engineering practice, and oftentimes, we try to be cost-effective. So we do not deal with high-order terms. Oftentimes, it just happens with linear functions. In one-dimensional situations, you have just a straight line. In two-dimensional, you have a planar representation. So linear function is just a name. So really, in two-dimensional space, you have this xy relationship. In three-dimensional relationship, and then you have the planar function. So you can just expand into higher and higher dimension. So these are still mathematical concepts. As engineers, we tend to think of an object, so a system. The system, we can just mathematically summarize the behavior of the system with an operator, say L, or O, or anything you want to call. So this is an operator. The functionality of the system is such, whenever you have an input, V, then we see an output, W. So mathematically, we say, OK, suppose V is a function of time, and the W is also a function of time. So the V of t going through the system changes a little bit, and it becomes W of t. So this is a concept of a system. So to be brief, so we just drop the dependency on time. So we say W equal to operator L upon input V. So V and W are one-dimensional function in the time domain. But the concept is not limited to one-dimensional function. And the V can be, say, a picture, high-dimensional picture, a tensor, or just a discrete object. It can be anything, just like a mathematical function. Remember, I said, you have an input domain. You have an output range. And in the two sides, you can have real number. You can have a complex number as an input. Or you just have a discrete side. So you have, say, RGB. The output is a certain mixture. So you can have all kinds of things. And you can think of myself as a system. And the student asks a good question, so I feel happy. So the happiness is the output of many things. So this is just so general. So the system, as engineers, we tend to think our job is to do system engineering. We always want to do work systematically. And we deal with a specific object, like a CT scanner. So we think more in terms of system. And to deal with the system relationship, then we use mathematical relationship. Instead of mathematical research, in mathematical field, you always deal with the mathematical functions. Just think of the abstract formulas and the relationships and so on. So as engineers, we see we have many systems to work with. And the first of all, we need to do measurement. And before we can measure, we really cannot do anything. So measurement is fundamentally important in engineering. And also in physics, according to quantum mechanics, so measurement makes a lot of sense. Without measurement, you do not know what's going on. You do not know cat is dead or alive. You do measurement, then the probability stators collapse into a reality. You know what's happening. So when you do measurement, you have input variables. And you could have a reference, like you use your weight. You need to keep balancing. So you put a certain weight on one side, then you product on the other side. If it is in good balance, then you know the weight of your product must be equal to the standard weight. So this is a signal, like a reading, say, is 1 kilogram or 2 kilogram. And then you do electrical signal measurement. So you have this meter. And you can have some standard voltage as a reference. And then you can change to different options to make sure the dynamic range of resistor that is to be measured. So you can modify the system. Like in this case, you just decide the range of resistance, or current, or voltage. So you have a lot of options. So the measurement example is an ohm meter, or some universal meter. You can measure direct current, alternative current, resistance, capacitance, inductance, and so on. All good examples. So this is just an electrical engineering example. As a more general example, like instructor. Instructor, so the system is myself. So whenever I see my calendar is input, then I just come to office. So you have this video, audio stream that's output. The modifier, say, I got sick. I couldn't come. So this system will work differently. I got an invitation. I have to go out to Boston, deliver lectures, and just cannot do the same thing. So this is many kinds of possibilities. And the imaging system is also a special kind of measurement system. So what you measure is a cross-sectional image, like MRI scanner is a system. It's an instrument called a scanner. So whenever you have idealized input, that is a cross-section of your physiological or anatomical status. So you have the gold standard, ground truth here. And suppose the ground truth is a single point. So this is the output, very soft, just a single point. Another system, not as good as the system one. The single point look a little blurred, pretty much like a camera. You can have a very high resolution camera, so very bright, small dot. We will show as a soft, high contrast dot. And another cheaper camera, the small dot look a little blurred. So if the system blurs a small point in this special case, and it will normally blur the whole image, this is what you got. This blurring function is a key characteristic of the system. And if you have a good scanner, then the output will be clearer. So this is an example talking about MRI scanner imaging system, the sensory system, imaging system, measurement system, basically just in the same category. So yeah, good science, any system somehow performs the measurement. And you just deal with input in a different way. So control system, you have input, then you have output. And the tricky thing is that the output will be feedback to effect the input, so that the system can behave in a more desirable way. A good example is a radar. So you use a radar to detect, to measure airplane. So airplane keep moving, so the signal feedback, the radar will be steered in a way so that the target may remain in the field of view. So this is a control system. And many complicated control systems, such as robotic system, now is a very hot topic. And people are talking about to develop robotic systems, such as in hotel, or just help take care of a baby, or just in nursing home. So a lot of wonderful opportunities. So this is a special control system with a certain level of machine intelligence, so that you can just perform a job adaptively. And a more complicated system, like a neurological system. So we have an important organ called the brain. So many, many neurons in it. Neurons take input from the environment, then decide if the neuron will be fired or not. They will send the signal down and generate intelligent activities. So neurological system, and more generally, physiological system altogether is so complicated. It's a multilevel system, from genomic level all the way up to physiological unit, organs, and the different systems, like a circulation system, the gestation system, and so on. So whole human system is so complicated. When something wrong, we are in diseased status. And if we want to treat a disease, we have to have a systematic point of view. That is called systems biology, or systems medicine. Systems biomedicine. So this is a huge area of research. And the robotic system and the human system, eventually, I believe, will merge in something like brain and machine interfaces. So we can have something implanted in the brain. We can just have more memory and reasoning power. So that's just the future. Convergence of robotic machine intelligence and human intelligence, neuroscience, and artificial intelligence, those things are very important. I remember I listened to a Google talk and read their articles talking about the most important thing, neuroscience and physics. Physics is about outside world. Neuroscience is about our internal world. And those two things merged together will generate a new kind of human being, eventually. So this is about the concept of systems and the many examples. And obviously, we do not have time and energy and knowledge to deal with all kinds of systems. And we narrow down to what we call linear system. What is a linear system? This is just the most important type of system. And coincidentally, medical imaging modalities all can be treated as a linear system. It's easy, but still a lot of things are going on. We need to learn. And once we have a good understanding of linear systems, and we are prepared to deal with nonlinear systems, just like you have to learn linear functions before you learn nonlinear functions. Nonlinear always order of magnitude more complicated. So still, we have a system with operator L. You have input V. You have output W. And the width is operator upon V. So this output. And this is just a general system concept. How the general system becomes a linear system is simple. It's important. So you see this right diamond again. So you need to have additivity and homogeneity, the two properties. If these two properties are satisfied, then we call the system linear system. So what is additivity? So suppose you have V1. Here you have W1. You have V2. You have W2. So these two inputs, V1 and V2, can be combined together. So I gave you the facts, two facts. You have V1 as input. You have W1 as output. This is fact one. Fact two, you have V2 as input. Output is W2. Then the question is, what if I give you third input, which is sum of V1 and V2? The simple thing you can imagine, if you add V1 and V2 together on the input side, then the output is simple, W1 plus W2. Very simple. So this is one of the key features of a linear system. We call it additivity. So the operator L is additive. So isn't that very simple? So this is the first requirement. Second requirement, I still give you two facts. Fact one, so you have input V, and then you have output W. And second, let's just give you one fact like this. Then I ask you, what if I scale V? By scaling, I mean, so the V is multiplied by a scalar called alpha, a scalar alpha. So then what will be output? Again, very naturally, the simple thing. So what would you expect? And then you would start with a simple thing. The simple guy will be, OK, the output will be alpha times W. So output will be scaled the same way. That's a simple answer. You could have a simpler answer. Just say no matter the output is 0. That's even simpler. But to be meaningful, reasonably consistent, and we say we have the homogeneity. So you have a scalar alpha, and the output will be scaled, will be multiplied, modified the very same way. So that is the case. And we call L is homogeneous. So these two properties are the key. And we know if the system is linear, you must have these two properties. So if the system is linear or not, oftentimes it can be judged very simply. For example, if a system is linear, then what will be the output of input

0? So output will be 0. So this is very simple from the homogeneity property. So you have alpha being 0, so the whole thing is 0. Alpha is outside here, so 0 times W will be 0. So you have this. So you use a simple testing point. You know if the system is linear, then you give 0 input, output must be 0. Then we say if this system is linear or not, you immediately say this is not linear. Why you put 0 as input, what is output? Output is Y, so the system is not linear. So things like this trick will help you decide, in some cases, the system is linear or not. But if you have this property, does not necessarily mean the system must be linear. Just like a function going through the point 0, 0 on the XY plane, it could be linear function, could be non-linear function. But this is a trick for you to know. And also, see the additivity. You can apply additivity multiple times. So you have V1, you have V2, V3, keep going, you have V100. So if you add all these 100 inputs together, the output will be, if the system is linear, output will be this. It will be summation of the output from W1 all the way to W100. And you may intend to extend to infinity the main input. Add it together, the summation, it changes. So the upper bound is infinity. And will this also be true? Be careful. You need to make sure when you involve infinitely many numbers, you sum them together. You need to make sure the summation converges. Otherwise, it doesn't make sense. So there are some rigorous requirements there. Anyway, so much for the linear system concept. Let me give you two figures to help you remember the two properties. So first is additivity. So as I told you, do not worry about notations. You can use any notation, because we are talking about general concepts of system, linear system, functions, linear functions, and so on. So you have input F1. So the output is K1. So this is F1, for example, two-dimensional picture. Then the output is a rotated version. So you have a certain mapping, certain transform. So the system does nothing, just turn it around. So you got this output. Then you have second input, F2. The system does the same rotation. So you got output here. The input is F1 and F2 superimposed together. You have a green cross. The green cross you put into the system. The system really do not care what you gave to him. The system just turn it around. So you do the same operation. So it doesn't matter you turn it together, or you turn it individually, then add them together. So this is illustration about additivity. About the homogeneity. So you have the input. So additivity, I give you two facts, I ask one question. Homogeneity or scaling property, I give you one fact. So then I ask you a question. So the fact is that you have F1, you got K1. So what if I scale F1? So I time the function of F1 with a number, say alpha 1. So this will modulate the amplitude of the function. So the image look brighter. So you got this one. What will happen to the output? The output picture or image will be scaled the same way, will become brighter to the same degree. So this is called homogeneity. So this is the concept about linear system. We need to have a crystal clear idea about linear system. And you are, if you know additivity and homogeneity, these two requirements are satisfied by a system. So sounds simple so far. Let me give you some exercise, some creed. And you'll have three functions or systems. The first one, map x, positive x to square root x. Is this linear or not? Put a number, x equal to 0 is 0, so that's not an issue. But if this system, by definition, it's not that you have zero functional system output when you have zero input. That's not definition. That's just a property, derived property. The definition really two properties, additivity and homogeneity. So let me ask you, if this system is linear or not, anyone want to try? So this does not satisfy additivity. And it does not satisfy homogeneity either. So it's not linear system. This square root, you have the curve, it's not linear. Suppose you have a vector, x. You can think x have three components, x, y, z. This is an arbitrary point in the three dimensional space. Then we perform a linear transformation. So the operator L is a 3 by 3 matrix A. Then you do the matrix multiplication. That's a linear transformation. An arbitrary point in three dimensional space through the linear transform A times vector x moves to another point. So you can show this system really satisfy the additivity and the scaling or homogeneity property. This is a linear system. And the integral derivative, these are also linear operators. You can show, give a function, I have x got this. You have f1, f2 as input. And then the integral will be f1 plus f2. And then you can separate it. And then you can show integral is a linear operator. So this is pretty much the concept of linear system. If you just look at regular textbook or appendix, it can explain linear system, then two lines, additivity, homogeneity, all the two things put together. They call it superposition principle. So you have a scalar alpha. You have v1, v2, a scalar alpha 1, alpha 2. Superposition principle says if input is alpha 1 times v1 plus alpha 2 times v2, then output will be alpha 1 times w1 plus alpha 2 times w2. This is superposition. This is easy. It's just derived directly from additivity and homogeneity. So you just see two lines as that. But now, let me see. As a researcher, I always would like to check all the details. And I remember once, many years ago, a short article, they say the linear system definition really redundant. You do not need two properties. You can really just need one property. The other can be derived from the first property. That is to say, you just ask the system satisfy additivity. Then you can derive homogeneity. All the systems satisfy homogeneity. You have a scaling vector, a scaling scalar. You can use the scalar scale. That's called homogeneity. And from homogeneity, you can really derive the additivity. So in that case, the definition, when you do definition, it's not an easy thing. Not arbitrarily, I say I define. You do definition, it's something like creation. You've got to be smart. If the definition is not well done, then you ask two properties. Actually, they are just interdependent from one you can derive the other. Why not you just define in a way so that you just ask any system that satisfy additivity? That's called the linear system. Why bother you say two things? So the definition is something very important, very critical. It's somehow like postulation somehow in some way. Euclidean geometry, the pioneers, Euclidean, and he defined the whole system from five postulates. And he is so much admired. He is able to see all the geometrical fact. Then he separated them and just put it in the system. And he was able to identify five fundamental properties that he called postulates. And from the five postulates, you can derive all the theorems and lemma and so on. So these are five postulates, so important. Then the question was asked, if these five postulates are independent or not? The fifth postulate is so important. And people feel maybe the fifth postulate could be derived from first four postulates. So a lot of effort, genius, spent time. Later on, it was realized that the fifth postulate is independent. Depends on different postulates. In the place of fifth postulate, you could have traditional flat Euclidean distance or Euclidean space. Or if you change a little bit of the fifth postulate, you have curved space. And Einstein's relativity really works in curved space. So this really shows in a way. So the independence and the fundamental postulation or definition, very important. So in the next few slides, I will show you some examples on what condition the additivity can derive homogeneity and the other way around. Just to give you two simple examples. And also, I give you some examples to show you this additivity and that the system does not have homogeneity. So you cannot go from left to right. And the fourth example is to show you. You can have a homogeneity, and you cannot go back to additivity. So this part represents a deeper thinking. I put a green button. So as a BME student, some of you majored in biology, chemistry, so those things may not be needed. But I think these are good things to think. And RPI education does not just give you some recipe. And we really want to train your mind. You think deeply. And while other people feel granted, and you see problems, you can explore for deeper linkage. And some inspiration could come out. So we can do better. Remember our slogan, to change the world. You need to think differently. We'll give you 10 minutes' rest. And then we come back, talking about these relationships. Let's continue. So this is an interesting slide to show you the relationship. Let me just give you some simple argument to say that in the case of continuous function, like you have continuous one-dimensional function, and higher-dimensional functions can be similarly preceded. So in the continuous case, additivity and homogeneity, these two properties are equivalent. So to show the equivalence, so we should do two things. First, you show if you have additivity, you can move from red box to blue box. So you have this orange arrow, so covered here. Then you can go the other way around, from blue box, homogeneity, go back to the right one, additivity. So if you have this, you can derive that. If you have that, you can derive this. So if you show these two ways work effectively, then we say the two things mean each other. So one means the other. So they're just equivalent. But to show if you have additivity, you can derive homogeneity is a small quiz. So how you prove that? So I already gave you answer. For additivity, you think you have v1 as x, v2 is also x. You have two x as input. You combine them together. Output will be f of x plus f of x. That will be 2x, right? You have this. Generally, you have this. This integer number, n times x, use additivity. You got this one. So this is a scaling property, the homogeneity. So the integer, positive integer number can be factorized outside. But the scalar, usually we like to have real scalar. So let's just say in this case, continuous case, we assume the scalar is real. So you have integer part satisfies the homogeneity property. What about, say, fractional, like a rational number? Rational number can always be represented as n divided by m. And the n and the m are positive integers. So we just consider if the homogeneity holds for real positive fractional number. And look at this. x is nothing but m times x divided by n. m is a positive integer. As I argued in the first line, that can be factored out here. So this means you can really move this m to the other side. So that means if you divide m both sides, so this side and this side, divided by m. So that means 1 over m times fx is equal to fx of x over m. So this is interesting. So that means the scalar here, the scalar is 1 over m. So this scalar inside 1 over m can be moved out. So you combine this one and this one, you know the scalar can be n over m. So that is a rational number, positive 1. And also, we know that this is a linear system, because we are talking about a linear system. So linear system, so you have a 0 element, and x minus x is equal to 0 by additivity. So this f0 equal to f of x plus f of minus x. That means f of minus x equal to minus fx. This is from here, because they say f of 0, I told you, is 0. So if f of x plus f of minus x equal to 0, then f of minus x must be minus fx. So by this relationship, you can extend the fractional positive scalar to the negative domain. So all these arguments are putting together. That means the scalar can be any fractional rational number. Positive or negative, it will work. Then the scalar inside is a real number. Now you know the scalar can be any rational number. Real number includes rational numbers and irrational numbers. And the rational numbers are very dense. And the irrational numbers are also very dense. But the function is continuous. So if the function is continuous and we know the scaling property or homogeneity holds for all densely distributed rational numbers, it must hold for the rational number. Because this is continuous. This is the key. So by that argument, this yellow box shows that this yellow arrow must be valid. Now let's look at the other way around. If you have homogeneity, and how you can prove you also have an additivity. Suppose you pick up an element called x, which is not 0. x equal to 0 can be easily handled. So the proof I gave here is not complete. It's just a key point. And I didn't show what if x equal to 0. Then x equal to 0 is easy to handle. You can show the argument holds true for x equal to 0. Suppose x is not 0. It's not 0, then any two different elements in the input domain, x1, x2. And we can scale, find the scalar. So it's alpha 1, alpha 2. So x1 equal to alpha 1 times x. x2 equal to alpha 2 times x. And look at this. What is the result of f of x1 plus x2? And we want to show f of x1 plus x2 equal to f of x1 plus f of x2. If we can show that, that means we have proved additivity. So why this is true, we see. So x1 plus x2 becomes alpha 1x plus alpha 2x because of the definition here. And then we know the homogeneity property holds. So this factorizes as x times alpha plus alpha 2. The sum of alpha 1, alpha 2 is a single scalar. That scalar can be factorized outside the function f of x because of homogeneity. So you've got this one. You've got this one, then you distribute it. You've got these two. Because the homogeneity holds, so this alpha 2 can be put back as alpha 1x for the first term. Alpha 2 can be put back inside the argument of function f. So you've got this one. This is nothing but f of x1, and this is f of x2. So we show the equivalence. You follow me? Follow me? So this is not complicated. So here is something you may not realize, but this could be something scary. What do we mean by scary? So what if you present two things, like additivity and homogeneity, and each of them look very reasonable. And later on, they do not play together very well. That means your system is not consistent. So a modern mathematics topic, a theme, is to show the whole modern mathematics system is consistent. That means you have postulate. You have theorem. After different procedures and all violated logical reasoning procedures, they never produce self-contradicting results. Then you show the modern system of mathematics is self-consistent. And this is not an easy thing. And it's still open question. Somehow, you define an animal by gene editing. And you play the role of God. You define certain things. And you hope it will work well. But if they do not play together, then they have some problem, internal problem. So that's not an easy thing. And the mathematical system ought to be self-consistent. So you cannot define certain things later on. And they are not consistent. So one looks very evident, reasonable thing after many, many steps, derive something self-contradicting or in contradiction with other properties you define. So doing creation, job, or definition, build a formal system is not easy. You ought to make sure to be consistent with your thought of the whole process. It's not easy. There are some mathematical logic theory that show for any formal system. And you cannot show the system is self-consistent, within the system itself. So there are some very profound thought. But anyway, this is definitely beyond the scope of our class. So now we move on talking about really, in certain cases, they are equivalent. But in other cases, they are not equivalent. So one property may carry some definite information, not contained by the other property. For example, in the following example, additivity does not mean homogeneity. So you do need two properties to qualify a system, a linear system. So for example, we're talking about a congregate operation for complex number. The complex number z, you'll have two components, real part and the imaginary part. The complex number is a really wonderful creation by mathematicians. So you'll have this z operation called congregating operation. So you really reflect this number with respect to real x's. So this z equal to x plus iy is mapped to this z congregate equal to x minus iy. So the plus becomes minus, just a reflection. So this is input, and this z star is output. This is a function, operation, or system. I use the words interchangeably. So that means if you have function fz defined as a congregating operation, then you have the congregating, by definition, congregating operation. So you've got output, say the z equal to a real part plus ib imaginary part. So as the output of the system or function, you've got a minus ib. This is clearly additive, because you have two numbers added together, g1 plus g2. So you do congregate for the sum. Then you can, by definition of congregating operation, you can do upon each component. So each component, by definition, is f of g1 plus f of g2. So f of g1 plus g2 becomes sum of individual functional values. This is nothing but the additivity. So that shows this operation, this system, satisfies the additivity. However, this system does not satisfy homogeneity. Why? Again, it's simple. You just put a complex scalar. The scalar, last time, I used a real scalar. But here, let's use a complex scalar. And again, the congregating operation, through the distribution, you've got this one. And this is generally not the same as alpha times z star. Alpha star times z star is not, in general, equal to alpha z star. So what's amazing here is z star is not z star. They are congregating numbers each other. And only if the scalar is on the real axis, the star or not star, make a difference. Because they are mirroring each other, unless they stay on the horizontal axis. So this is nothing but alpha times f of z. So alpha times f of z is not equal to f of alpha z. So homogeneity is not the case. It shows this case. And now, let me just show the last case. If you have homogeneity, and it's not necessarily true, that you also have additivity. So here, I just copy a paragraph from my book draft. So let's define a real-valued function as follows. This is a linear function, just a case-wise linear function. It's not a piece-wise. It's case-wise. So in case y, suppose x is rational. So again, we view the number axis as a collection of rational and irrational numbers. So if a number x is a rational, then you have a slope called my. If the number is a rational, you have a slope m2. And the m1 and the m2 are not the same. So this is a function I defined. And clearly, it makes sense. And they give me a number. I first check if it's rational or not. Then I apply a different slope. And that's just a functional value map that is clear. And so if we're talking about a scalar in the rational domain, I only use a rational scalar. A scalar is something important. You cannot just talk about scalar. You have to talk about what is a scalar domain. It's an integer. It's a rational number, real number, complex number. You need to make sure, just like I said, I have a vector. You are not totally clear. You would ask me to clarify. Is your vector two-dimensional, three-dimensional, likewise? So here, I say the scalar is a rational number. Clearly, I say in this case, homogeneity is not an issue. Because whether the number is a rational number or irrational number, and the number times a rational number, and the property is not changed. So if you have a rational number times a rational number, the result is still a rational number. If you have an irrational number times a rational number, the result is still a rational number. So that's not a problem. So you do the scaling. Then you can still use the same slope. Why is there M1 or M2? Because multiplication with a scalar will not change the property of the number. So the same slope can be used in either case. It's a linear function. So the scaling of homogeneity property is there. So not a problem. The problem is that you have the homogeneity. But from homogeneity, you cannot generally reach the additivity. And here, let me just give you a single counterexample. Suppose you have two irrational numbers, and the x1 and the x2. And these two numbers are purposely picked up so that their sum is a rational. So can you consider a case you have an irrational number, one irrational number, the second irrational number. They're added to each of them. Each of them is irrational. But they're added together as a rational. Any number you can figure out. This is some simple example. You can imagine you have a number, say, square root 2, irrational number. The other number, 100 minus square root 2, is still an irrational number. You add it together, you get 100. That's a rational number. So in this case, if you add them together, the result is because you have an irrational number you add it together. So it becomes a rational number. So a rational number times m1, that's not equal to m2 times the result. The m2 times the sum can be distributed and expressed as f of x1 and f of x2. Because this x1, x2 individually, they are irrational numbers. So they should use slope m2. But if they sum together, they are rational numbers. They should use slope m1. And these two slopes, we already told you, they are not the same. So you do not have additivity in this case. So these are subtleties. So you need to understand why we really should say linear system must have, in general, must have both additivity and homogeneity. You cannot just like all the article discussions and they think definition was not well done. And you only need one property, and the rest will be derived. It's not the case. So the key part already mentioned, so this is what I mentioned to you. The additivity and the homogeneity can be summarized as a so-called superposition principle. Superposition principle showing the green text is equivalent to the two properties, additivity and the homogeneity. So we deal with a linear system. And we have some very powerful tools. Like later on, we will explain to you Fourier analysis, convolution. And linear system much simpler than nonlinear system. But still, not easy. You have some hard core science to learn how you deal with a linear system. And luckily, any nonlinear system piecewise, and you can use nonlinear system can be approximated as many linear systems. And just like an arbitrary function, you can use piecewise linear approximation. So once you know linear system theory and technologies, and then you are well prepared to deal with nonlinear system. And the further comment, and I would like to call it relative linearity. So the linearity always show that linear system always has a property. You have a zero component, the output is zero. That's by definition. And just like linear function, you have when the intercept is zero. So you have x equal to zero, y will be equal to zero. But a general linear function, you can have a non-trivial intercept. So when x equal to zero, y can be b, b is intercept. So that's still a linear function. And whether the intercept is b not zero or b equal to zero is kind of subjective. It depends on how you select the coordinate system. So the requirement I use, the trick I told you, to verify a system linear or not, you put a zero element c, if output is zero. This is not very important, I think, because the coordinate system can be arbitrarily selected. So a better definition, I think, you focus on relative change. So you have a system. The system may have a certain initial status, like a certain b, like a linear function. But if you select the coordinate system in the right way, just make b equal to zero, then you turn an arbitrary linear function into a linear function with b equal to zero. And that is the linear system. So in this case, we say suppose the system has initial status. We don't care about the initial status. Just like a linear function, we don't care about intercept. Just assume there's something there. Then we go from there on. We see the relative change. So with respect to that initial status, you have initial change, delta xy. Then you just observe what is the change in output. That's delta y1. And you have delta x2. You have delta y2. The superposition property holds for the relative change. So the relative changes, delta x1, delta x2 linearly combined. Then the relative changes will be combined linearly in the same way. So we are talking about additivity and homogeneity in terms of relative change. So as long as these two properties are valid in terms in the relative sense, then we think the system is linear. So this is a good way to convert some by original definition. It's not a linear system. But you just change the coordinate system. Then those nonlinear systems will become linear systems. And then we can use linear systems theory. And in electrical engineering and the classical components, you have three kinds of components, the resistor, capacitor, inductor. And the voltage-current relationship satisfy almost all. So it's a linear function. So you have V equal to

I times R. If current is 0, V is 0. This is a linear system. The other way I equal to factor 1 over R times V is a linear system. So that's straightforward, linear function, linear system. Now in electrical engineering, oftentimes we need to use a capacitor. A capacitor has the voltage and the current relationship described by integral operator. And this integral operator may have a initial term. So that means V depends on current change and also depends on initial status at the time t0. So you have a relative change. The V will be changed accordingly. So if you just check the first line, you have this intercept, which in general is not 0. This is not a linear system. But if we think in relative terms, so the system will be intrinsically linear. If the change in input and the change in output, they are good enough satisfying identity and homogeneity. So this is one case. And the integral part can be converted into partial derivative, not partial derivative, into a differential relationship. This is linear, just a fit of original case. And the sister component of capacitor is inductor. So the Vi, the current and the voltage relationship, and just the other way around. So V is related to first derivative. Here, I is related to first derivative. So these are just, I'm not sure in high school you learned these or not. My son in high school in robotic program, they learned these things, but not so much about calculus. They used some verbal description. So just use your calculus knowledge. Believe me, this is a relationship. So V is linked to current change. This is a linear relationship with 0 intercept. But in terms of I, the current passing through the coil is dependent on the initial status, and also the voltage change over the coil. So this initial current is initial status. And the system can be dealt with using linear system theory in terms of relative changes. So this is what I would like to explain. And there are several paragraphs in the textbook you can read if you care to understand more. So now let me go a step further about the linear system. What's a linear system? You say any system satisfies both the additivity and the homogeneity. Or equivalently, you can say any system that satisfies the superposition principle, that is a linear system. Then you get a full mark. You know linear system definition. But linear system as so defined is still very complicated. And we want to just simplify a little further. And also, the simplification makes sense. And the medical imaging system, major medical imaging modalities are not only linear system. They are also safely invariant linear system. This is a very rough idea. What do you mean safely invariant? Let me give you an example. You are in the classroom. I take a picture of you. This is just your input, the iPhone is a system. The picture is output. Then you meet me in my office hour. And I take a picture of you in my office. And again, input is you and output is iPhone picture. They are the same. What's different? And you shifted from this big classroom to my small office. Your picture remains same. So that is safety invariability. So this is oftentimes the case shows the system has a stable personality. And your credibility means you count you. Today you told me you will do good work and I count on. So you will not change. So safety invariability can be with respect to space, with respect to time, and respect to other features. So this is very important. You need to know this concept really well. So what do I mean by safety invariance mathematically? So let me give you an example. Again, this is a linear system and operator L. So you have input V, output W. So it's got the same thing here. And there's an additional requirement called safety invariability, safety invariant. Then you have this input. The input is delayed in time by a constant amount of tau. What is output? Output will be delayed by the same amount of time. Like my lecture, last year I talked about linear system. A year ago, so tau equal to a year. My teaching system is roughly linear safety invariant. I changed a little bit, but the essential content is still the same. So not a perfect safety invariant system. I keep improving. So this is a concept. And certainly, the formulation is for one-dimensional function. You can think about vectors and other input element, and vectors and matrix multidimensional functions. And as long as the shift happens in the input domain by a certain amount, and the output will be sifted in the same amount of time, space, or anything relevant. Another example is a water pond. You have a water drop. It will generate a ripple like this. The wave is increasing. So you've got the wave shape. You just extract this. You see this ripple. It's just radically symmetric. So you got the output when you input to the pond system at this particular point. But what if you sift the input point from here to here? Then you extract the same thing happened. So this is basically an invariant linear system. And you listen to music in a different way. You have this small device. You just rotate the handle. So you've got nice music come out. Or you just price your iPhone music. You've got a certain sum of your favorite. You do it today. You enjoy the music. Very, very happy. You do it a week later, same music, similar feeling. So that's a temporary invariant system, music playing system. Or this music box, sound box, the same thing. So let me just explain a little further. So you have this system represented as a box. You have input f of x, like this. This is the output. But now input is changed by amount of a. So f of x minus a. So output, because the system, I told you, the system is sift invariant. So the output will be g of x minus a. So this point, x equal to 0, is moved to this position, x equal to a. Because when x equal to a, you do the subtraction. This is 0. So you've got a point here. So this point corresponds to this point. Likewise, every point will be sifted by the amount of a. So this waveform just is shaped uniformly towards the right-hand side by the amount of a. So this sift happens in the output domain. So this waveform is also sifted by the same amount of a. Why? Because this is a sift invariant system by definition. There's so much you need to know about system, about linear system, and about sift invariant linear system. And the sift invariant system can be linear, can be nonlinear. Nonlinear system can also be sift invariant. Anyway, so last topic, and you see this green button. It's just something nice to know. And that is about nonlinear system. Nonlinear science is relatively new and very complicated. So nonlinear equations are difficult to derive, to find a closed-form solution. But with a modern computer, you can just discretize the equation. And many times, you just do a brutal force computation. You can still get meaningful results. And more interesting than the practical application is that nonlinear system generates some amazing mystery phenomena, which are often counterintuitive or unpredictable. So anything counterintuitive by definition is very interesting. Let me give you some simple example. Still have about 10 minutes. So I'll give you some simple example. This is so-called logistic map. But this thing may, there is a nature paper. You know, the best journal, most reputable journal, are two, science and nature. And this is just one slide that really summarizes one nature paper published decades ago. It's a very simple nonlinear relationship. So the input is xn. Output is xn plus 1. Just one line, very simple. This is nonlinear because the linear system would be xn plus 1 equal to r times xn. r is slope. That's a linear system. Here, you have a nonlinear system simply modified. We simply modify the linear function by a single simple term. That is 1 minus xn. So just do this. So if xn is very small, so this factor will be small. The system is kind of linear. But if xn is not very small, it's a quadratic function. Quadratic function is certainly nonlinear. So this is a mathematical definition. But what does this relationship mean? There are some real meanings. Say the xn represents the number of certain animals, like a rabbit, this year. Then we want to fit into this formula. We predict what will be the number of rabbits next year. So let's just do this prediction. The meaning is that if you don't have many rabbits in forest, the rabbit number will grow linearly. This is a probability. But you have too many rabbits. The wolf, or so on, get excited. The resources are not enough. So this is just a constraint for the growth of the animal population. So when this number gets small, it goes linearly. When this number is a little bit larger, this will reduce the rate of increase. So this is the kind of model widely used to study population growth. And also, in a forest, presumably, you have a maximum number the forest could take. Say this small forest cannot take one billion rabbits. So the reasonable maximum number can be used to normalize this equation, so that just use normalize the population, kind of a percentage related to full capability. So the percentage becomes a number between 0 and 1. 1 means you've got to reach full potential, full capability, cannot be more, so cannot grow anymore. So this factor becomes 0, cannot grow anymore. So it's just overcrowded, so all things will be broken down. So this is a relationship. So x normalizes the population. n means current year. n plus 1 means next year you do prediction. r is the growth rate, so very reasonable. Very reasonable like this, you may expect that you have a certain animal number, say 500. Next year, 800. 800 kind of reach the upper limit. So next year, you've got to buy, because just too many rabbits and not enough food, and just different competing factors. Next year, number buy. So this is the guy to say 500 again. And then 500, next year, spring, you have a lot of good things. You go back to 800, you just got a stable erotic solution. And the real result, so when the number is small, you do have a single number, say always about 600 rabbits. Or you have, say, 500 and then 800, 500 and 800, or 300 and then 1,000, just oscillating from two numbers. When the x, when the number, when the mean the growth rate. When the growth rate, for certain growth rate, you oscillate, you have a stable solution. For other growth rate, you're oscillating between two numbers. But when the growth rate is high, the higher and higher, so you have many numbers to jump between. And when the number is quite big, like you got close to four, so it seems you've got a random number. So you cannot predict what will be next year's population of rabbits. And you see the number as a function of time jumping around. So you cannot predict what is the exact number. So you kind of have a random number this way. So this random number is not a purely random number. It is governed by a deterministic relationship. So kind of randomness from deterministic relationship is called chaotic behavior. So chaotic science is very important, like a wiser forecast. And we show the system is a chaotic system. That means it's very hard to do long-term forecasting. And you click this MATLAB link. You can download or just copy it from my slides. Just a few number, just a few lines. And you keep changing R. You can just reproduce this figure on your computer screen. So this is a nonlinear system. It exhibits chaotic behavior. And the interesting idea is that the relationships in the world oftentimes are nonlinear. So once you have chaotic behavior, the question is you cannot predict the outcome precisely. Even small perturb, like a butterfly, just perturb a little bit, it will affect the long-term behavior, while the final outcome lands depending on small random factors. So even you know the relationship, if you do not have total control of the environment, you'll have some fluctuation, random things. You cannot do long-term prediction reliably. And it goes back to fundamental physics. Quantum mechanics really shows the fundamental description is probabilistic. That means the long-term precise prediction is impossible in principle. So this is the philosophical meaning of chaotic science. And the last example I want to show you, neuron and human neuron is a nonlinear system. So you have different stimuli into the neuron. If the stimuli is not very strong, the neurons stay there. Just like you said there, you'll have some small stimulator, like a very light wind that you don't have. But when this input accumulates to a certain degree, like your fellow students punch you, then you immediately react, you fight back. So the neuron accumulates input. If the accumulated input is not strong, no reaction. But once it's over a threshold, it will send the electrical potential, called the neuron is fired on. So electrical signal will move on. This is a nonlinear system. You have a threshold. The threshold is used to deal with noise flow. So it's small noise, I ignore. But if you go way above the noise level, I react. So we can mathematically model biological neurons. We call it artificial neurons. So you have multiple input with different weight. You sum weighted input together as a single number. So this weighted sum, called inner product. This is a linear part. Then this summation results is nonlinearly converted to output. It depends on threshold. Below threshold, the input is very small. Over threshold, all of a sudden, the output will be high. So this is another example of nonlinear system. This neuron can be connected into different architecture, called a neural network. You can have many, many layers, called a deep neural network, or deep learning method. Deep neural network is, right now, the main pros for deep learning or machine learning. So lower layer, input layer, and some hidden layer give you local features. And the higher and higher layer, you recover facial expression. Then this can help recognize students, or passengers, or traffic sign, auto driving technology. Use deep neural network, a good linear system. So just show math for your general understanding. I mentioned that there are three examinations in the homework. And those of you, particularly motivated, want to do some class project, it's just the opposite of thing. You can talk to me. But I recommend that you look into machine learning. And you can read my perspective article. And if you are interested, you can talk to me. Otherwise, you just do regular classroom teaching, examination, homework combination. So today's homework, a TA will upload. So you can just download the general format. Our last slide is homework. And then you'll have assignment today. A week later, like by midnight, next Tuesday, you'll upload. The TA will grade. And then after deadline, we will post answers. And these are open questions, just for you to think about. That's all for today. Thank you.